**Lectures 3-4: Consumer Theory**

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**Consumer Theory**

Consumer theory studies how rational consumer chooses what bundle of goods to consume.

**Rational** has a new meaning now: unbounded rationality!

Special case of general theory of choice.

Key new assumption: choice sets defined **ONLY** by prices of each of goods, and income (or wealth).

**Consumer Problem (CP)**

**Restrictions that don’t show up are the most important ones!**

Notation: , , (inner product)

* Consumer chooses consumption vector
* is consumption of good
* Each unit of good costs
* is total expenditure
* Total available income is

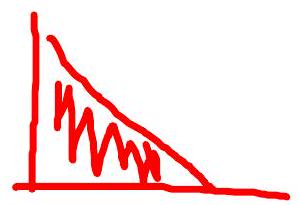
Now discuss some implicit assumptions underlying (CP).

**First: Prices are Linear**

Each unit of good costs the same.

No quantity discounts or supply constraints.

Consumer's choice set (or budget set) is:



Set is defined by single line (or hyperplane): the budget line

Assume .

(If we’re talking about bad things, so that price cannot be positive, think of garbage collection with positive price)

**Second: Goods are Divisible**

and consumer can consume any bundle in budget set

Can model indivisibilities by assuming utility only depends on integer part of .

**Third: Set of Goods is Finite ()**

This is not obvious: think of a dynamic economy with no known final date.

Debreu (1959): *A commodity is characterized by its physical properties, the date at which it will be available, and the location at which it will be available*.

In practice, set of goods suggests itself naturally based on context.

**Marshallian Demand**

The solution to the (CP) is called the Marshallian demand (or Walrasian demand).

May be multiple solutions, so formal definition is:

**Definition**: The Marshallian demand **correspondence** is defined by

Heavy notation for simple idea!

Domain is : prices, one level of income.

Start by deriving basic properties of budget sets and Marshallian demand.

Example: Cobb-Douglas Marshallian demand:

**Budget Sets**

**Theorem**: Budget sets are homogeneous of degree 0 : that is, for all , .

Proof:

Nothing changes if scale prices and income by same factor. QED.

**Theorem:**

If , then is compact.

“Proof” (Write it formally as an exercise)

A subset of is compact if and only if it is closed and limited.

For any is closed. (Notice the weak inequality in the definition of .)

If , then is also bounded.

QED.

**Marshallian Demand: Existence**

**Theorem:**

If is continuous and , then (CP) has a solution.

(That is, is non-empty.)

Proof.

A continuous function on a compact set attains its maximum (Weirstrass theorem). QED.

**Marshallian Demand: Uniqueness?**

The Marshallian Demand needs not be unique

Example: perfect substitutes – write it out as an exercise.

Generally, the Marshallian demand is a correspondence, or a set-valued function: for each , it associates a set of optimal choices .

We’ve seen before the following results (we’ll just rewrite them in our context):

Since the budget set is convex, the Marshallian demand is a convex correspondence if preferences are convex.

The Marshallian demand is unique (that is, a function) if preferences are strictly convex.

**Marshallian Demand: Homogeneity of Degree 0**

**Theorem**

For all .

Proof:

, so (CP) with prices and income is same problem as (CP) with prices and income , since utility function is not affected by QED.

**Marshallian Demand: “Continuity”**

**Theorem**

Assume is continuous and strictly quasi-concave (that is, preferences are strictly convex). Then the Marshallian demand is a continuous function.

More generally: if is continuous (but not necessarily quasi-concave), then the Marshallian demand is upper hemicontinuous.

Proof: follows directly from the theorem of the maximum.

**Marshallian Demand: Walras' Law**

**Reminder:** Preferences are locally non-satiated if for all and all , there exists such that .

**Theorem**

If preferences are locally non-satiated, then for every and every , we have .

Proof:

If , then there exists such that . By local non-satiation, for every there exists such that .

Hence, there exists such that .

But then , that is, is not an optimal choice: contradiction. QED.

Walras' Law lets us rewrite as

**Marshallian Demand: Differentiable Demand**

Implications if demand is single-valued and differentiable:

* A proportional change in all prices and income does not affect demand:
* A change in the price of one good does not affect total expenditure:
* A change in income leads to an identical change in total expenditure:

**The Indirect Utility Function**

Can learn more about set of solutions to (CP) (Marshallian demand) by relating to the **value** of (CP).

Value of welfare of consumer facing prices with income .

The value function of is called the indirect utility function.

**Definition**

The indirect utility function is defined by

So we have

Notice that demand is the image of and also the argument of .

Example: Cobb-Douglas indirect utility function:

**In our context, the Theorem of the Maximum implies**:

Marshallian demand is upper hemicontinuous (if it’s a function, it’s continuous)

Indirect utility is continuous

**Indirect Utility Function: Properties**

**Theorem**

The indirect utility function has the following properties:

1. Homogeneity of degree 0 : for all , .
2. Continuity: if is continuous, then is continuous on .
3. Monotonicty: is non-increasing in and non-decreasing in . If and preferences are locally non-satiated, then is strictly increasing in .
4. Quasi-convexity: for all , the set is convex. (Consumer is worse off at average prices/income.)

**Proof**:

1. Follows from homogeneity of degree zero of Marshallian demand.
2. Follows directly from the Theorem of the Maximum.
3. Left as an exercise.
4. Pick two elements in the domain of : and

Assume:

Define

We need to show that

We will show something stronger: for all such that , one has .

Use the definition of :

This holds if and only either of .

Then we have:

In any case, , as we wanted to show.

QED.

Interpretation of quasi-convexity: consumer prefers extreme prices/income than average ones.

Extreme prices allow consumers to explore substitution: something desirable must be cheap enough.

(Income is one dimensional and hence it follows immediately that the average of two income levels is lower than the highest of them.)

GRAPHIC

**Indirect Utility Function: Derivatives**

When indirect utility function is differentiable, its derivatives are very interesting.

Q: When is indirect utility function differentiable?

A: When is (continuously) differentiable and Marshallian demand is unique.

**Theorem**

Suppose (1) u is locally non-satiated and continuously differentiable, and (2) Marshallian demand is unique in an open neighborhood of with and . Then is differentiable at .

We’ll skip the proof. For details if curious, see Milgrom and Segal (2002), "Envelope Theorems for Arbitrary Choice Sets."

Or check chapters 3 and 4 in Stokey and Lucas (1989).

Furthermore, letting , the derivatives of are given by:

and

where is any index such that .

* Suppose consumer's income increases by .
* Should spend this dollar on any good that gives biggest "bang for the buck."
* Bang for spending on good equals : can buy units, each gives utility .
* Finally, for precisely those goods that maximize bang for buck.
* marginal utility of income equals , for any with .

**Indirect Utility Function: Derivatives**

* Suppose price of good increases by .
* This effectively makes consumer poorer.
* Just saw that marginal effect of making poorer is , for any with .
* marginal disutility of increase in equals , for any with .

**Kuhn-Tucker Theorem**

**Theorem (Kuhn-Tucker)**

Let and be continuously differentiable functions (for some ), and consider the constrained optimization problem

If is a solution to this problem (even a local solution) and a condition called constraint qualification is satisfied at , then there exists a vector of Lagrange multipliers such that

and

**Kuhn-Tucker Theorem: Comments**

1. Any local solution to constrained optimization problem must satisfy first-order conditions of the Lagrangian
2. Condition that for all is called complementary slackness.

* Says that multipliers on slack constraints must equal 0.
* Consistent with interpreting as marginal value of relaxing constraint .

1. There are different versions of constraint qualification. Simplest version: vectors are linearly independent for binding constraints.

Exercise: check that constraint qualification is always satisfied in the when , and preferences are locally non-satiated.

**Lagrangian for**

For two goods:

Generally:

is multiplier on budget constraint.

is multiplier on the constraint .

FOC with respect to :

Complementary slackness: if . So:

What’s the intuition of ?

Implication: marginal rate of substitution between any two goods consumed **in positive quantity** must equal the ratio of their prices .

In other words: slope of indifference curve between goods and must equal slope of budget line.

Intuition: equal "bang for the buck" among goods consumed in positive quantity.

**Back to Derivatives of**

When is differentiable, we have the following result:

**Theorem**:

Proof:

For the first part:

Without loss of generality, take only two goods: , .

Then:

Take the derivative with respect to income :

But . Rewrite the previous equation:

We also know that **for all** . This allows us to differentiate both sides with respect to , and get:

This is exactly the term in square brackets in the previous equation, which becomes:

In short:

For the second part:

This holds because we always have (Kuhn-Tucker).

Differentiate with respect to :

Collect terms to get:

QED.

These are applications of the envelope theorem: ignore indirect effect of changes in parameters (that is, impact through changes in optimal decisions).

**Envelope Theorem**

Theorem (Envelope Theorem)

For , let be a differentiable function, let

, and let

.

If is differentiable at then, for any ,

**Back again to Derivatives of**

Combining with if , obtain

for any with .

This proves above theorem on derivatives of .

We've already seen the intuition.

**Roy's Identity**

Under conditions of last theorem, if then

**Key Facts about (CP), Assuming Differentiability**

* Consumer's marginal utility of income equals multiplier on budget constraint: .
* Marginal disutility of increase in price of good equals .
* Marginal utility of consumption of any good consumed in positive quantity equals .

**The Expenditure Minimization Problem**

In (CP), consumer chooses consumption vector to maximize utility subject to maximum budget constraint.

Also useful to study "dual" problem of choosing consumption vector to minimize expenditure subject to minimum utility constraint.

This expenditure minimization problem (EMP) is formally defined as:

**Hicksian Demand**

Hicksian demand is the set of solutions to the EMP.

The expenditure function is the value function for the EMP:

is income required to attain utility when facing prices .

Each element of is a consumption vector that attains utility while minimizing expenditure given prices .

Hicksian demand and expenditure function relate to EMP just as Marshallian demand and indirect utility function relate to .

**Exercise**: find the Hicksian demand and the expenditure function for the Cobb-Douglas utility function.

**Why Should we Care about the EMP?**

For this course, 2 reasons:

(1) Hicksian demand useful for studying effects of price changes on "real" (Marshallian) demand.

In particular, Hicksian demand is key concept needed to decompose effect of a price change into income and substitution effects.

(2) Expenditure function important for welfare economics.

In particular, use expenditure function to analyze effects of price changes on consumer welfare.

**Hicksian Demand: Properties**

**Theorem (MWG 3E3)**

Assume , preferences are locally non-satiated, and . Then the Hicksian demand satisfies:

1. Homogeneity of degree 0 in : for all , .
2. No excess utility: if is continuous and , then for all .
3. Convexity/uniqueness: if preferences are convex, then is a convex set. If preferences are strictly convex and "no excess utility" holds, then contains at most one element.

Proof:

1. Minimizing or yields the same result for any . QED.
2. The proof is by contradiction.

Assume by contradiction that at the solution, .

Take , for .

The continuity of implies that for close enough to one, , and .

That is, it is possible to find some that respects the constraint and decreases expenditure. Contradiction. Hence one cannot have at the solution.

QED.

1. Left as an exercise.

**Expenditure Function: Properties**

**Theorem (MWG 3E2)**

The expenditure function satisfies:

1. Homogeneity of degree 1 in : for all , .
2. Continuity: if is continuous, then e is continuous in and .
3. Monotonicity: is non-decreasing in and non-decreasing in . If "no excess utility" holds, then is strictly increasing in .
4. Concavity in p: e is concave in .

Proof:

1. For all , we know from the previous proposition that .

Then one may write:

1. Follows from the Maximum Theorem.
2. Show first that is strictly increasing in

The proof is by contradiction.

Assume by contradiction that is not strictly increasing in . That is:

Let and be optimal to achieve utility levels and , respectively.

Assume and . This is the contradiction.

Build a new bundle: for .

is continuous implies that there is some close enough to one such that:

Then is not optimal to achieve .

Let’s show now that is non-decreasing in .

Take two price vectors , such that , and, for all , .

Let be optimal for prices . Then:

The first inequality follows from the previous line: .

The second inequality follows from the definition of .

It follows that is non-decreasing in .

QED.

1. To show concavity, fix some level .

Define for some .

Let be optimal for . Then:

The inequality comes from the definition of and from the fact that .

QED.

**Intuition for concavity:**

Start with and an optimal bundle .

If prices change to but is fixed, new expenditure is : linear in .

If consumer may adjust to minimize , new expenditure cannot be larger.



**Expenditure Function: Derivatives**

Shephard's Lemma: if Hicksian demand is single-valued, it coincides with the derivative of the expenditure function.

**Theorem**

If is continuous and is single-valued, then the expenditure function is differentiable in at , with derivatives given by

Intuition: If price of good increases by , unique optimal consumption bundle now costs more.

**Proof:**

Recall that

Given that is differentiable in , envelope theorem implies that

**Comparative Statics**

Comparative statics are statements about how the solution to a problem change with the parameters.

(CP): parameters are , want to know how and vary with and .

(EMP): parameters are , want to know how and vary with and .

Turns out that comparative statics of (EMP) are very simple, and help us understand comparative statics of (CP).

**The Law of Demand**

"Hicksian demand is always decreasing in prices."

Theorem (Law of Demand)

For every , and , we have

Example: if and only differ in price of , then

Hicksian demand for a good is always decreasing in its own price.

Graphically, budget line gets steeper shift along indifference curve to consume less of good 1.

Proof:

By definition:

Add these inequations:

Factor out and :

Change for :

Now factor out :

This is the law of demand.

QED

**The Slutsky Matrix**

If Hicksian demand is differentiable, can derive an interesting result about the matrix of price-derivatives

This is the Slutsky matrix.

A symmetric matrix is negative semi-definite if, for all .

**Theorem**

If is single-valued and continuously differentiable in at , with , then the matrix is symmetric and negative semi-definite.

Proof.

Follows from Shephard's Lemma () and Young's Theorem.

**The Slutsky Matrix**

What's economic content of symmetry and negative semi-definiteness of Slutsky matrix?

Negative semi-definiteness: differential version of law of demand.

Ex. if with 1 in the component, then , so negative semi-definiteness implies that .

Symmetry: derivative of Hicksian demand for good with respect to price of good equals derivative of Hicksian demand for good with respect to price of good .

Not true for Marshallian demand, due to income effects.

**Relation between Hicksian and Marshallian Demand**

Approach to comparative statics of Marshallian demand is to relate to Hicksian demand, decompose into income and substitution effects via Slutsky equation.

First, relate Hicksian and Marshallian demand.

Let’s ask a simple question: **what’s the Marshallian demand evaluated at some exogenous utility level ?**

At first, this seems nonsense: Marshallian demand is a function of prices and income , not of utility .

In fact, utility is not even exogenous: it is endogenous when one solves for the Marshallian demand, which is computed exactly maximizing utility.

Yet, we can evaluate at any level of income … that is, at any monetary value .

So we can choose in particular , since is a monetary value: it is the minimum expenditure to reach the exogenous utility level .

Hence and , so we can write : Marshallian demand as a function of some exogenous utility level .

Next question: what’s the Marshallian demand evaluated at ?

Intuition is simple: it is simply the Hicksian demand: .

Analogously, we may write : Hicksian demand as a function of income.

Then .

So these equalities hold for the solutions to UMP e EMP.

A similar reasoning applies to the value functions of these problems:

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If is the most utility consumer can attain with income , then consumer needs income to attain utility .

If need income to attain utility , then is most utility consumer can attain with income .

The next result formalizes these relationships.

**Theorem** Suppose is continuous and locally non-satiated. Then:

1. For all and and .
2. For all and and .

Proof: left as an exercise.

**The Slutsky Equation**

**Theorem (Slutsky Equation – MWG 3G3)**

Suppose is continuous and locally non-satiated. Let and . If and are single-valued and differentiable, then, for all ,

Proof:

For all ,

Since this holds for all goods and **for all prices**, one may differentiate both sides with respect to some price :

But we know that . Substitute into the previous equation to get:

QED

Intuition: If increases, two effects on demand for good :

* Substitution effect:
* Movement along original indifference curve.
* Response to change in prices, holding utility fixed.
* Income effect:
* Movement from one indifference curve to another.
* Response to change in income, holding prices fixed.

**Terminology for Consumer Theory Comparative Statics**

Definition

Good is a normal good if is increasing in . It is an inferior good if is decreasing in .

Definition

Good is a regular good if is decreasing in . It is a Giffen good if is increasing in .

Definition

Good is a substitute for good if is increasing in . It is a complement if is decreasing in .

Definition

Good is a gross substitute for good if is increasing in .

It is a gross complement if is decreasing in .

**Comparative Statics: Remarks**

* Both the substitution effect and the income effect can have either sign.
* Substitution effect is positive for substitutes and negative for complements.
* Income effect is negative for normal goods and positive for inferior goods.
* By symmetry of Slutsky matrix, is a substitute for is a substitute for .
* Not true that is a gross substitute for is a gross substitute for .
* Income effects are not symmetric.